CORRELATION BETWEEN THE SOLUTIONS OF THE GENERALIZED AND CLASSICAL THERMOMECHANICS OF DEFORMED SOLIDS

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The article establishes a correlation beween the solutions of boundary problems of classical and generalized heat conduction, and also between the solutions of problems of various sections of the thermomechanics of deformed solids.

Detailed investigation of the processes connected with heat transfer requires consideration of more-complex laws of heat conduction than Fourier's law. This, in particular, is necessary for avoiding the paradox of infinite speed of propagation of heat [1, 2]. The simplest generalization of the classical equation of heat conduction eliminating this paradox is the hyperbolic equation [3]

$$k\nabla^2\Theta - c\dot{\Theta} - \tau_0 c\ddot{\Theta} = -w - \tau_0\dot{w}. \tag{1}$$

It is a consequence of the law of conservation of energy and of the law of heat conduction in the following form:

$$\mathbf{q} + \tau_0\dot{\mathbf{q}} = -k\nabla\Theta. \tag{2}$$

If we take into account the dependence of the free energy on the history of change of the temperature field, and of the heat flux on the history of change of the temperature gradient, we obtain an even more complex equation [4, 5]. In this case the law of heat conduction is written as follows:

$$\mathbf{q} = -\int_0^t K(t-\tau)\nabla\dot{\Theta}(\tau)\,d\tau, \tag{3}$$

and the equation of heat conduction has the form

$$\int_0^t \left\{K(t-\tau)\nabla^2\Theta(\tau) - \frac{\partial}{\partial t}C(t-\tau)\dot{\Theta}(\tau)\right\}d\tau - C(0)\dot{\Theta}(t) + w(t) = 0. \tag{4}$$

It should be noted that Eq. (4) can be reduced to Eq. (1) if the relaxation functions are

$$C(t) = c,\quad K(t) = k\{1 - \exp(-t\tau_0^{-1})\}. \tag{5}$$

Many authors dealt with the solution of the boundary problems of generalized heat conduction and thermoelasticity. A detailed bibliography is contained in [6, 7]. The authors of [8-10], worked out a fairly general method of solving a certain class of such problems, and Korol'kov and Pupin [11] suggested an apparatus of special functions which is very useful in the transition from Laplace transform to the originals in problems of generalized heat conduction.

Let us examine some boundary problems for Eq. (4). Assume that Θ inside the region bounded by the surface O satisfies Eq. (4), and on the surface the boundary conditions of one of three types apply:

$$\Theta|_O = f(t),\ \mathbf{q}\cdot\mathbf{n}|_O = g(t),\ \mathbf{q}\cdot\mathbf{n} + H\Theta|_O = h(t). \tag{6}$$

We assume zero initial conditions. Using the integral Laplace transform, we obtain from (3), (4) the following equations in the image space:

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$$\mathbf{q}_p = -pK_p(p)\nabla\Theta_p(p), \tag{7}$$
$$p^2C_p(p)\Theta_p(p) - pK_p(p)\nabla^2\Theta_p(p) - w_p(p) = 0.$$

In that case the boundary conditions assume the form

$$\Theta_p|_O = f_p,\ \mathbf{q}_p\cdot\mathbf{n}|_O = g_p,\ \mathbf{q}_p\cdot\mathbf{n} + H\Theta_p|_O = h_p. \tag{8}$$

We note that the classical law and equation of heat conduction after the Laplace transform are written as follows:

$$\mathbf{q}_p(p) = -k_0\nabla\Theta_p(p),$$
$$pc_0\Theta_p(p) - k_0\nabla^2\Theta_p(p) - w_p(p) = 0. \tag{9}$$

If we compare (7) and (9) we can easily see that the solution of the boundary problem for Eqs. (7) can be obtained from the solution of the boundary problem for Eqs. (9) if we replace in the latter c_0 by $pC_p(p)$ and k_0 by $pK_p(p)$.

Consequently, we use the procedure of constructing the solution of the boundary problem of the generalized theory of heat conduction if the solution of the corresponding problem for the classical equation of heat conduction is known. The known solution has to be subjected to a Laplace transform, in it c_0 has to be replaced by $pC_p(p)$, and k_0 by $pK_p(p)$, and from the obtained transform the original has to be found. It should be noted that this method of constructing a solution may be used for finding the exact as well as an approximate solution.

The system of equations of symmetric thermoelasticity, which is based on the law of heat conduction (2), has the form [6]

$$\mu\nabla^2\mathbf{u} + (\mu+\lambda)\nabla\nabla\cdot\mathbf{u} - \nu\Theta_0\nabla\vartheta + \mathbf{X} = \rho\ddot{\mathbf{u}}, \tag{10}$$
$$k\nabla^2\vartheta - \tau_0 m\Theta_0\ddot{\vartheta} - m\Theta_0\dot{\vartheta} - \tau_0\nu\nabla\cdot\ddot{\mathbf{u}} - \nu\nabla\cdot\dot{\mathbf{u}} = -\Theta_0^{-1}(\tau_0\dot{w} + w).$$

It can easily be seen that here we have a change of parameters that converts, in the space of the transforms, the solution of the boundary problem of classical bound thermoelasticity into the solution of the boundary problem of generalized thermoelasticity. In this case, in the known solution of the classical problem after the Laplace transform, k_0 has to be replaced by $k(1 + p\tau_0)^{-1}$. This, of course, also applies to the theory of asymmetric thermoelasticity [12], where such a change also establishes a correlation between the solutions.

It is known that in the space of Laplace transforms there exists a correlation between the solutions of the theory of symmetric elasticity and viscoelasticity [13]. Smirnov [14] found a change of parameters establishing a correlation between the solutions of the generalized asymmetric thermoviscoelasticity and classical bound thermoelasticity of Kosser's medium.

Thus, the above-explained method of constructing the solution is of a fairly general nature and makes it possible to use the large stock of known solutions of boundary problems of the classical theory. Of course, such a way of constructing the solution is not always optimal. If the established correlation is to be used, the problem has to be reduced with the aid of a partial solution to a problem with zero initial conditions.

In addition to solving actual problems, the above correlation between the solutions may also be used for investigating theoretical problems. In particular, with its aid the potentials of a number of sections of the thermomechanics of deformed solids may be constructed. For instance, the results obtained by different methods in [15, 16] may be easily substantiated by the existence of the above correlation between the solutions, and the potentials of the equation of heat conduction with memory were constructed in [17], in fact, by using this correlation.

The mentioned correlation between the solutions of problems of various sections of thermomechanics establishes a correspondence between them in the space of the Laplace transforms. An analogous correlation in the theory of viscoelasticity is established by the correspondence principle [13, 18]. However, this general correlation between the solutions is considerably simplified in some cases and may be expressed as the correlation between the solutions in the space of the originals (quasistatic problem of viscoelasticity, the problem of steady-state harmonic viscoelastic oscillations [18]).

In analogy with this, a correlation can be established in some cases directly between the solutions of the generalized and the classical problem also in generalized thermomechanics. We will examine problems of steady-state harmonic vibrations of the generalized asymmetric thermoelasticity. The volume V, bounded by the surface O, is acted upon by the forces, moments, and sources of heat release of the form

$$\mathbf{X} = \mathbf{X}_0 \exp(-ist), \quad \mathbf{Y} = \mathbf{Y}_0 \exp(-ist), \quad w = w_0 \exp(-ist), \tag{11}$$

and the boundary conditions on the surface O express an analogous dependence of the specified values on time. In that case the problem reduces to the construction of the solution of the system obtained in [12], which, in accordance with the assumption (11), is written as follows:

$$\begin{aligned} &(\mu+\alpha)\nabla^2\mathbf{u}_0 + (\mu-\alpha+\lambda)\nabla\nabla\cdot\mathbf{u}_0 + 2\alpha\nabla\times\boldsymbol{\omega}_0 - \nu\Theta_0\nabla\vartheta_0 + \mathbf{X}_0 = -\rho s^2\mathbf{u}_0,\\ &(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}_0 + (\gamma-\varepsilon+\beta)\nabla\nabla\cdot\boldsymbol{\omega}_0 + 2\alpha\nabla\times\mathbf{u}_0 - 4\alpha\boldsymbol{\omega}_0 + \mathbf{Y}_0 = -s^2\mathbf{I}\cdot\boldsymbol{\omega}_0,\\ &k\nabla^2\vartheta_0 + im\Theta_0 s(1-is\tau_0)\vartheta_0 + i\nu s(1-is\tau_0)\nabla\cdot\mathbf{u}_0 = -\Theta_0^{-1}(1-is\tau_0)w_0. \end{aligned} \tag{12}$$

However, the solution of the classical problem of steady-state harmonic oscillations of a thermoelastic Kosser's medium reduces to the construction of the solution of a system of the form

$$\begin{aligned} &(\mu+\alpha)\nabla^2\mathbf{u}_0 + (\mu-\alpha+\lambda)\nabla\nabla\cdot\mathbf{u}_0 + 2\alpha\nabla\times\boldsymbol{\omega}_0 - \nu\Theta_0\nabla\vartheta_0 + \mathbf{X}_0 = -\rho s^2\mathbf{u}_0,\\ &(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}_0 + (\gamma-\varepsilon+\beta)\nabla\nabla\cdot\boldsymbol{\omega}_0 + 2\alpha\nabla\times\mathbf{u}_0 - 4\alpha\boldsymbol{\omega}_0 + \mathbf{Y}_0 = -s^2\mathbf{I}\cdot\boldsymbol{\omega}_0,\\ &k_0\nabla^2\vartheta_0 + im\Theta_0 s\vartheta_0 + i\nu s\nabla\cdot\mathbf{u}_0 = -\Theta_0^{-1}w_0. \end{aligned} \tag{13}$$

A comparison of systems (12) and (13) shows that to obtain the solution of Eqs. (12) it suffices to replace k_0 by $k(1 - is\tau_0)$ in the solution of system (13).

As an example of utilizing the correlation between the solutions in this form, we will examine the complex potential of the problem of steady-state harmonic oscillations of classical thermoelasticity. The potential of a simple layer in this case [19], accurate to the designations, has the form

$$\begin{aligned} &\mathbf{P}(\mathbf{r}) = \iint_O \{\boldsymbol{\chi}(\mathbf{r}_0)\cdot[\Lambda(R)\mathbf{E} + \nabla_0\nabla_0\Phi(R)] + \chi_0(\mathbf{r}_0)\nabla_0\Gamma(R)\}\,dO,\\ &P_0(\mathbf{r}) = \iint_O \{-is\boldsymbol{\chi}(\mathbf{r}_0)\cdot\nabla_0\Gamma(R) + \chi_0(\mathbf{r}_0)\Psi(R)\}\,dO, \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\Lambda(x) = \frac{1}{2\pi\mu x}\exp\{ix\eta_3\},\\ &\Phi(x) = \frac{\mu+\lambda}{2\pi\mu(2\mu+\lambda)x}\sum_{n=1}^{3} M_n \exp\{ix\eta_n\},\\ &\Gamma(x) = -\frac{\nu}{2\pi(2\mu+\lambda)k_0 x}\sum_{n=1}^{n} L_n \exp\{ix\eta_n\},\\ &\Psi(x) = \frac{1}{2\pi k_0\Theta_0 x}\sum_{n=1}^{2}\left(\eta_n^2 - \frac{\rho s^2}{2\mu+\lambda}\right)L_n \exp\{ix\eta_n\}, \end{aligned} \tag{15}$$

and χ and χ_0 are the potential densities. Here the following notation is introduced:

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{\frac{1}{2}},$$

$$\nabla_0 = \mathbf{e}_x \frac{\partial}{\partial x_0} + \mathbf{e}_y \frac{\partial}{\partial y_0} + \mathbf{e}_z \frac{\partial}{\partial z_0}, \tag{16}$$

$$L_1 = -L_2 = (\eta_1^2 - \eta_2^2)^{-1},$$

$$M_n = (\eta_n^2 - \eta_0^2)(\eta_n^2 - \eta_{n+1}^2)^{-1}(\eta_n^2 - \eta_{n+2}^2)^{-1}, \ \eta_4 = \eta_1, \ \eta_5 = \eta_2,$$

$$\{x_0, \ y_0, \ z_0\} \in O.$$

Then η_1 and η_2 are the roots of the biquadratic equation

$$\eta^4 - \eta^2 \frac{k_0 \rho s^2 + is\Theta_0 [\nu^2 + m(2\mu + \lambda)]}{(2\mu + \lambda) k_0} + \frac{i\rho s^3 m \Theta_0}{(2\mu + \lambda) k_0} = 0, \tag{17}$$

and

$$\eta_0 = \frac{is\Theta_0}{(\mu + \lambda) k_0} [\nu^2 + m(2\mu + \lambda)], \quad \eta_3 = s\sqrt{\rho\mu^{-1}}. \tag{18}$$

The roots η_1 and η_2 are chosen in such a way that

$$\operatorname{Im}(\eta_1, \ \eta_2) > 0. \tag{19}$$

It is perfectly obvious that in accordance with what had been explained, expression (14) is the potential of a simple layer of the problem of steady-state oscillations of generalized thermoelasticity if Eq. (17) is replaced by the following equation:

$$\eta^4 - \eta^2 \frac{is\Theta_0 (1 - is\tau_0)[\nu^2 + m(2\mu + \lambda)] + \rho k s^2}{k(2\mu + \lambda)} - \frac{i\rho s^3 \Theta_0 m (1 - is\tau_0)}{k(2\mu + \lambda)} = 0, \tag{20}$$

the constant η_0^2 is put equal to

$$\eta_0^2 = \frac{is\Theta_0 (1 - is\tau_0)}{(\mu + \lambda) k_0} [\nu^2 + m(\mu + \lambda)], \tag{21}$$

and the functions contained in (14) are replaced by

$$\Gamma(x) = -\frac{\nu(1 - is\tau_0)}{2\pi\mu(2\mu + \lambda) kx} \sum_{n=1}^{2} L_n \exp\{ix\eta_n\}, \tag{22}$$

$$\Psi(x) = \frac{(1 - is\tau_0)}{2\pi k \Theta_0 x} \sum_{n=1}^{2} \left(\eta_n^2 - \frac{\rho s^2}{2\mu + \lambda}\right) L_n \exp\{ix\eta_n\}.$$

The function $\Lambda'(x)$ retains its form, and in the function $\Phi(x)$ only the constants M_n and η_n change in accordance with the replacement of Eq. (17) by (20). In exactly the same way, on the basis of the classical potentials [19], we can construct another four potentials of the problem of steady-state harmonic oscillations of generalized thermoelasticity.

NOTATION

Θ temperature; k, thermal conductivity; c, specific heat; w, heat release density; τ_0, magnitude characterizing the speed of propagation of heat; **q**, heat flux vector; K, C, relaxation functions; **n**, vector of the external normal to the surface O; p, parameter of Laplace transform; **u**, displacement vector; ω, vector of rotation; ρ, density of material; **I**, tensor characterizing inertial properties of the material in microrotation; ν, magnitude characterizing the thermoelastic properties of the material; μ, λ, α, γ, ε, β, elastic constants; $\vartheta = (\Theta - \Theta_0)\Theta_0^{-1}$, relative deviation of the temperature from the initial one; **E**, unit tensor.

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